RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2017

SECOND YEAR [BATCH 2015-18]

Date : 18/05/2017

MATHEMATICS (Honours)

Time : 11 am – 3 pm

Paper : IV

Full Marks : 100

[Use a separate Answer Book for <u>each group</u>]

<u>Group - A</u>

Answer **<u>any five</u>** questions from **<u>Question Nos. 1 to 8</u>**: [5×7]

1. a) Suppose $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$. Define $d:\mathcal{P}(\mathbb{N})\times\mathcal{P}(\mathbb{N})\to\mathbb{R}$ by, for $A,B\in\mathcal{P}(\mathbb{N})$,

$$d(A,B)=\begin{cases}0 \text{ if } A\Delta B=\phi\\ \dfrac{1}{m} \text{ if } m \text{ is the least element of } A\Delta B\end{cases}$$

Prove that 'd' is a metric on $\mathcal{P}(\mathbb{N})$. [5]

b) Give an example of a sequence $\{A_n\}$ of subsets of a metric space X such that $\overline{\left(\bigcup_{n=1}^{\infty}A_n\right)}\neq\bigcup_{n=1}^{\infty}\overline{A_n}$. [2]

2. a) If $f:\mathbb{R}\to\mathbb{R}$ is continuous then prove that $G(f)=\{(x,f(x)):x\in\mathbb{R}\}$ is nowhere dense in $\mathbb{R}^2$. [5]

b) Show that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued continuous functions over $\mathbb{R}$ then $\bigcup_{n=1}^{\infty}G(f_n)\underset{\neq}{\subset}\mathbb{R}^2$. [2]

3. a) Prove that l_∞ is not separable. [4]

b) Let A be a closed set in a metric space X. Show that there is a continuous map $f:X\to\mathbb{R}$ such that $A=\{x\in X:f(x)=0\}$. [3]

4. a) Prove that every countably compact metric space is pseudocompact. [3]

b) Suppose X is a metric space such that each continuous map $f:X\to\mathbb{R}$ is uniformly continuous. Show that X is complete. [4]

5. a) Prove that if X is a compact metric space then every open over of X has a Lebesgue number. [4]

b) Is total boundedness a hereditary property? Justify. [3]

6. a) Find an open cover of $[0,1]\cap\mathbb{Q}$ having no finite subcover. [3]

b) Use 5(a) above to show that if $f:(X,d_1)\to(Y,d_2)$ is continuous where (X,d_1) and (Y,d_2) are metric spaces and (X,d_1) is compact then f is uniformly continuous. [4]

7. a) Give an example of a bounded sequence in a metric space having no convergent subsequence. [3]

b) Suppose d_1 and d_2 are two different metrics on a set X and there exist $K_1,K_2>0$ such that $K_1d_1(x,y)\le d_2(x,y)\le K_2d_1(x,y)\ \forall x,y\in X$. Show that d_1 and d_2 are equivalent. [4]

8. a) Prove that $\mathbb{Q}$ is not connected. [2]

b) If $f:[0,1]\to[0,1]$ is a continuous map, show that f has a fixed point. [3]

c) Show that $C[a,b]=\{f:[a,b]\to\mathbb{R} \mid f \text{ is continuous}\}$ equipped with sup norm, is not a bounded metric space. [2]

Answer **any three** questions from **Question Nos. 9 to 13**: [3×5]

9. a) Let $r_1, r_2, r_3, \cdots$ be an enumeration of the set of all rationals in [0,1] and a sequence of functions $\{f_n\}$ is defined on [0,1] by $f_n(x)=0,\ x=r_1,r_2,r_3,\cdots,r_n$

$$=1, x\in[0,1]-\{r_1,r_2,\cdots,r_n\}$$

Show that $\{f_n\}$ is not uniformly convergent on [0,1]. [3]

b) Let $f_n(x)=x^n, x\in[0,1]$. Show that $\{f_n\}$ is not uniformly convergent on [0,1]. [2]

10. If a power series $\sum_{n=0}^{\infty} a_n x^n$ is neither nowhere convergent nor everywhere convergent, show that there exists $R > 0$ such that the series converges absolutely for all x satisfying $|x|<R$ and diverges for all x satisfying $|x|>R$. [5]

11. a) Show that $\sum_{1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n^2(1+x^{2n})}$ converges uniformly for all real x. [3]

b) Find the radius of convergence of $1-\frac{x}{1\cdot 2}+x^2-\frac{x^3}{2\cdot 4}+x^4-\frac{x^5}{4\cdot 8}+\cdots$. [2]

12. a) Prove that the series $\sum_{1}^{\infty} \frac{\cos nx}{\{\log(n+1)\}^x}$ is uniformly convergent on any closed interval [a, b] lying within $(0,2\pi)$. [3]

b) Find the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ where $a_0=1, a_n=\left(\sqrt[n]{n}+1\right)^n$, $n\geq 1$. [2]

13. Let g be a continuous function on [0,1] with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on [0,1]. [5]

Group - B

Answer **any three** questions from **Question Nos. 14 to 18** : [3×10]

14. a) Solve : $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$, given that $\left(x+\frac{1}{x}\right)$ is one integral. [5]

b) Solve : $x^2\frac{d^2y}{dx^2}-2(x^2+x)\frac{dy}{dx}+(x^2+2x+2)y=0$, by reducing it to normal form. [5]

15. a) Solve the equation : $3x^2\frac{d^2y}{dx^2}-(6x^2-6x-2)\frac{dy}{dx}-4y=0$ by factorisation of operators. [5]

b) Find all the eigen values and eigen functions of the Sturm-Liouville problem

$4\frac{d}{dx}\left\{e^{-x}\frac{dy}{dx}\right\}+(1+\lambda)e^{-x}y=0$ satisfying the conditions y(0) = 0 and y(1) = 0. [5]

16. a) Solve : $\frac{dx}{dt}-7x+y=0$, $\frac{dy}{dt}-2x-5y=0$. [4]

b) Write down the geometrical interpretation of the equations of the form $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, where P, Q, R are functions of x, y and z. [2]

c) Derive the necessary condition of integrability of the equation $Pdx+Qdy+Rdz=0$ where P, Q, R are functions of x, y and z. [4]

17. a) Form a partial differential equation by eliminating the arbitrary functions ϕ and ψ from $z=y\phi(x)+x\psi(y)$. [5]

b) Apply Charpit's method to find the complete integral of the equation $yzp^2=q$, $p\equiv\frac{\partial z}{\partial x}, q\equiv\frac{\partial z}{\partial y}$. [5]

18. a) Using Laplace transform, determine the solution of, $y''(t)+y(t)=t$, where $y'(0)=1$ and $y(\pi)=0$. [5]

b) Using Laplace transform, evaluate $\int_0^\infty e^{-tx^2}dx$, where t > 0. [5]

Answer **any two** questions from **Question Nos. 19 to 21** : [2×10]

19. a) Find the double point and discuss its nature for the curve $ay^2 = x^3$, a > 0. [4]

b) Find the radius of curvature of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at any point P(x,y) on it. Show that the radius of curvature at any point P(x,y) of the ellipse is bounded by the radius of curvature of the ellipse at the end points of the major and minor axes. [6]

20. a) Find the asymptotes of the curve : $3xy^2-2x^2y=a^2(x+y)+b^2$. [5]

b) Show that the pedal equation of the curve $c^2(x^2+y^2)=x^2y^2 (c\neq 0)$ with respect to the origin is $\frac{1}{p^2}+\frac{3}{r^2}=\frac{1}{c^2}$. [5]

21. a) If $u_n=\int_0^{\pi/2} x^n \sin x\, dx\ (n\geq 1)$ then show that $u_n=n\left(\frac{\pi}{2}\right)^{n-1}-n(n-1)u_{n-2}$. [4]

b) Find the area of the surface generated by revolving the cardioid $r=a(1+\cos\theta)$ about the initial line. [6]

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